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Finite correction to Aharonov-Bohm scattering by a contact potential

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Abstract. We study the influence of a contact (or delta) potential on the Aharonov-Bohm scattering of nonrelativistic particles. In general the contact potential has no effect on the scattering as expected. However, when the magnetic flux and the strength of the contact potential take some special values, the Aharonov-Bohm scattering cross-section is manifestly changed. It is shown that these special values correspond to the simultaneous existence of two half-bound states in two adjacent angular momentum channels. Two limiting processes are presented to deal with the singularity of the contact potential and results of the same nature are obtained.

PACS. 03.65.Nk Scattering theory – 03.65.Vf Phases: geometric; dynamic or topological

The Aharonov-Bohm (AB) effect [1] is one of the most remarkable effects in quantum theory. The theoretical predictions have been verified by experiments long ago [2,3]. Numerous works have been devoted to the study of AB effect in various fields of physics. The simplest aspect of this effect is the scattering of charged particles by a string of magnetic flux created by a very tiny solenoid or a magnetic iron whisker. In the idealized limit when the solenoid or the whisker is infinitely long and infinitesimally small, the scattering problem can be solved exactly in quantum mechanics. In the original work of AB and some subsequent ones the calculations are based on the Schrödinger equation [1,4–6]. Since the most frequently employed charged particles in experiments, the electron and the proton, both have spin 1/2, in later studies the Dirac equation is employed as a starting point. The calculations based on the Dirac equation showed [7,8] that the differential scattering cross-section for polarized particles is rather different from that obtained by using the Schrödinger equation even in the nonrelativistic limit. Recently the effect of the anomalous magnetic moment of the particle is considered and the Dirac-Pauli equation is employed [9]. It turns out that in most cases the results are the same as those without an anomalous magnetic moment. However, when the incident energy takes some special values, the cross-section for polarized particles is dramatically changed.

Another subject concerning the AB scattering is on the effect of some scalar potential. The hard core potential has been considered [4,5] to incorporate the effect of the device that generates the magnetic flux, namely, the solenoid or magnetic iron whisker. It was shown that the result reduces to the one of pure AB scattering when the radius of the hard core tends to zero. The inclusion of a two-dimensional Coulomb potential was also studied and some exact results was obtained in both nonrelativistic [10,11] and relativistic [12] cases. The latter model with both a vector AB potential and a scalar Coulomb one may approximately describe the relative motion of particles carrying electric charge and magnetic flux in two dimensions [11]. A circular ring potential (a delta function on a circle) or a contact potential (a delta function at the origin) has been considered in some literature of mathematical physics [13–15]. Such a model has been found to be useful in the study of semiconductor nanostructures [16] and persistent current in mesoscopic systems [17,18].

In this paper, we consider the influence of a contact potential on the AB scattering. The potential is of the form

$$V(r) = \frac{\Omega\hbar^2}{2M}\frac{\delta(r)}{r}, \tag{1a}$$

where M is the mass of the particle, Ω is a dimensionless parameter characterizing the strength of the potential, and r is one of the polar coordinates (r, θ) on the xy plane where the particle is moving. Since $r = 0$ is a singular point of the coordinate system, the above form of the potential is not well defined. Thus it is a formal expression of the $a \to 0$ limit of the circular ring potential

$$V(r) = \frac{\Omega\hbar^2}{2M}\frac{\delta(r-a)}{r} = \frac{\Omega\hbar^2}{2M}\frac{\delta(r-a)}{a}. \tag{1b}$$

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This will be employed in all practical calculations and the limit $a \to 0$ will be taken at the end. The difference of the above circular ring potential from that considered by previous authors [13–18] should be remarked. If a is finite, they are essentially the same. However, we are mainly interested in the case $a \to 0$ with Ω fixed. In this limit, our potential has a higher singularity. It is also remarkable that $\int V(r)\, d\boldsymbol{r} = \int_0^\infty V(r) 2\pi r\, dr = \pi\Omega\hbar^2/M$, so that when $a \to 0$ the potential is essentially the same as $(\pi\Omega\hbar^2/M)\delta(\boldsymbol{r})$, where $\boldsymbol{r}$ is the position vector on the xy plane. Therefore, the special form of the contact potential employed here is a rather natural one.

According to a previous work on the quantum scattering by a pure contact potential [19], in three dimensions (where r is a spherical coordinate) such a contact potential leads to a nonvanishing cross-section when $\Omega = -1$, while in two dimensions nontrivial results are obtained only when the contact potential involves an additional logarithmic factor. These results are somewhat surprising since according to classical mechanics the scattering cross-section for a contact potential is obviously zero. Anyway, the contact potential (1a) causes no scattering in two dimensions by itself. According to this conclusion and based on intuitive physical judgement, when combined with, say, the AB potential, the potential (1b) should give only small corrections when a is small, and such corrections should vanish when $a \to 0$. Although this turns out to be true in most cases, it is somewhat unexpected that the AB scattering cross-section is manifestly changed when Ω and the magnetic flux in the AB potential take some special values. We will show that these special values correspond to the simultaneous existence of two half-bound states [20] in two adjacent angular momentum channels. This is similar to the situation of a pure contact potential in three and one dimensions [19], where unexpected results for scattering also occur when there exists a half-bound state.

The inclusion of the above contact potential in the AB scattering may have physical interest in several aspects. First, it has been pointed out above that such models (with finite a) are useful in some condensed-matter-physics problems. Second, for particles carrying magnetic flux and electric charge, the charge-flux interaction may be described by the AB potential, and if the charge-charge interaction is of a very short range (since such particles in two dimensions are somewhat artificial, say, solitons in Chern–Simons field theory [21–28], various model potentials for the charge-charge interaction may be of theoretical interest), the Hamiltonian with both a vector AB potential and a scalar contact one may approximately describe their relative motion. Third, just as a hard core potential, a circular ring potential like (1b) may as well describe approximately the effect of the solenoid or the magnetic whisker itself. In this respect, however, there exist some difficulties if one attempts to examine the theoretical results by experiments. One difficulty is that one does not know what a device would correspond to the potential (1b) with the required value of Ω. Another difficulty is that the limit $a \to 0$ is difficult to realized in experiment, since in the following calculations it means $ka \ll 1$ (k is the wave number of the incident particle) or $a \ll \lambda$ (λ is the corresponding wave length). Anyway, the study may be of theoretical interest since it reveals an effect of point interactions that was not noticed before.

The vector AB potential is of the form

$$\boldsymbol{A}(\boldsymbol{r}) = A(r)\boldsymbol{e}_\theta, \tag{2}$$

where $\boldsymbol{e}_\theta$ is the unit vector in the θ direction. In the following calculations we take two different limiting processes. In the first process the magnetic field $\boldsymbol{B}(\boldsymbol{r}) = \Phi\delta(\boldsymbol{r})\boldsymbol{e}_z$ is concentrated at the origin, where Φ is the flux of the magnetic filed and $\boldsymbol{e}_z$ is the unit vector in the z direction. This corresponds to the choice

$$A(r) = \frac{\Phi}{2\pi r}. \tag{3a}$$

In the second process the magnetic field $\boldsymbol{B}(\boldsymbol{r}) = (\Phi/\pi a^2)\vartheta(a-r)\boldsymbol{e}_z$ is uniformly distributed inside the circle $r = a$, where ϑ is the step function. This corresponds to the choice

$$A(r) = \begin{cases} r\Phi/2\pi a^2, & r < a, \\ \Phi/2\pi r, & r > a. \end{cases} \tag{3b}$$

In both processes the scalar potential (1b) is employed and the limit $a \to 0$ is taken finally. We will see that the two processes give results of the same nature.

Now we write down the stationary Schrödinger equation

$$E\psi = H\psi = \left[-\frac{\hbar^2}{2M}\left(\nabla - \frac{iq}{\hbar c}\boldsymbol{A}\right)^2 + V(r)\right]\psi, \tag{4}$$

and consider partial wave solutions with positive energy $E > 0$:

$$\psi_m(r,\theta) = R_m(r)e^{im\theta}, \quad m \in \mathbb{Z}, \tag{5}$$

then we have the ordinary differential equation for the radial wave function

$$\frac{d^2R_m}{dr^2} + \frac{1}{r}\frac{dR_m}{dr} + \left[k^2 - \left(\frac{m}{r} - \frac{qA}{\hbar c}\right)^2 - \frac{\Omega}{a}\delta(r-a)\right]R_m = 0, \tag{6}$$

where $k = \sqrt{2ME}/\hbar$. Scattering solutions will be constructed from these partial-wave solutions.

Equation (6) should be solved separately for $r < a$ and $r > a$. The connection conditions at $r = a$ can be obtained by integrating the equation from $a-0$ to $a+0$. They are

$$R_m(a+0) = R_m(a-0),$$
$$R'_m(a+0) - R'_m(a-0) = \frac{\Omega}{a}R_m(a). \tag{7}$$

The solution for $r < a$ depends on the choice of $A(r)$, so it is different in the two processes. Since $A(r)$ has the same form for $r > a$ in both processes, the solution for $r > a$ is

$$\Delta f(\theta) = i\sqrt{\frac{2}{\pi k}}\left\{\sum_{m\geq\nu}\frac{e^{i\nu\pi}J^2_{m-\nu}(\xi)e^{im\theta}}{J_{m-\nu}(\xi)H^{(1)}_{m-\nu}(\xi)-2i/\pi\Omega}+\sum_{m<\nu}\frac{e^{i(m-\nu)\pi}J_{\nu-m}(\xi)[e^{-i\nu\pi}H^{(1)}_{m-\nu}(\xi)+e^{i\nu\pi}H^{(2)}_{m-\nu}(\xi)]e^{im\theta}}{2[e^{i(m-\nu)\pi}J_{\nu-m}(\xi)H^{(1)}_{m-\nu}(\xi)-2i/\pi\Omega]}\right\}. \quad (21)$$

the same. Note that the last term in the square bracket of equation (6) vanishes when $r \neq a$, we obtain the solution for $r > a$

$$R_m(r) = a_m\left[\cos\left(\delta_m - \frac{\nu\pi}{2}\right)J_{m-\nu}(kr) - \sin\left(\delta_m - \frac{\nu\pi}{2}\right)N_{m-\nu}(kr)\right], \quad r > a, \quad (8)$$

where a_m is an arbitrary constant, $J_{m-\nu}(kr)$ and $N_{m-\nu}(kr)$ are Bessel and Neumann functions, respectively, δ_m is the phase shift to be determined by the connection condition at $r = a$, and

$$\nu = \frac{q\Phi}{2\pi\hbar c}. \quad (9)$$

The above solution for $r > a$ is also valid for other scalar potential $V(r)$ that is vanishing beyond $r = a$, for example, a hard core potential.

We assume that the incident particles are from the left, then the scattering solution is

$$\psi(r,\theta) = \sum_{m=-\infty}^{\infty}\psi_m(r,\theta) = \sum_{m=-\infty}^{\infty}R_m(r)e^{im\theta}, \quad (10)$$

where the constant a_m in equation (8) is chosen as $a_m = i^m\exp(i\delta_m)$. The asymptotic form at $r \to \infty$ will be given below which shows that it is indeed a scattering solution.

First of all we write down the results for pure AB scattering. These are different from the original ones [1,6] in appearance since their incident particles are from the right. In this case $\Omega = 0$ and $R_m(r) = a_m J_{|m-\nu|}(kr)$ for the whole range of r. We denote

$$\nu = m_0 - \alpha, \quad m_0 \in \mathbb{Z}, \quad 0 \leq \alpha < 1, \quad (11)$$

then $R_m(r)$ can be written in the form of equation (8) with the phase shifts

$$\delta_m^{\rm AB} = \begin{cases}\nu\pi/2, & m \geq m_0,\\ m\pi - \nu\pi/2, & m \leq m_0 - 1.\end{cases} \quad (12)$$

By similar calculations to those of references [1,6], the asymptotic form of the scattering solution at $r \to \infty$ can be found to be

$$\psi_{\rm AB}(r,\theta) \to e^{ikx+i\eta(\theta)} + f_{\rm AB}(\theta)\sqrt{\frac{i}{r}}e^{ikr}, \quad (13)$$

where the scattering amplitude is

$$f_{\rm AB}(\theta) = i\frac{\sin\nu\pi}{\sqrt{2\pi k}}\frac{e^{i(m_0-1/2)\theta}}{\sin(\theta/2)}, \quad (14)$$

and the phase factor in the incident wave is

$$e^{i\eta(\theta)} = \exp\left\{im_0(\theta-\pi)+i\alpha\epsilon(\sin\theta)\left[\arcsin(\cos\theta)+\frac{1}{2}\pi\right]\right\}, \quad (15)$$

where ϵ is the sign function. Note that the phase factor is not continuous at $\theta = 0$ but is written in an explicitly single-valued form.

When $\Omega \neq 0$, the solution (10) can be written as

$$\psi(r,\theta) = \psi_{\rm AB}(r,\theta) + \frac{1}{2}\sum_{m=-\infty}^{\infty}i^m\left(e^{i2\delta_m} - e^{i2\delta_m^{\rm AB}}\right) \times e^{-i\nu\pi/2}H^{(1)}_{m-\nu}(kr)e^{im\theta}, \quad r > a, \quad (16)$$

where $H^{(1)}_{m-\nu}(kr)$ are Hankel functions. The asymptotic form at $r \to \infty$ is of a similar form to equation (13):

$$\psi(r,\theta) \to e^{ikx+i\eta(\theta)} + f(\theta)\sqrt{\frac{i}{r}}e^{ikr}, \quad (17)$$

where the scattering amplitude is

$$f(\theta) = f_{\rm AB}(\theta) + \Delta f(\theta), \quad (18a)$$

and

$$\Delta f(\theta) = -\frac{i}{\sqrt{2\pi k}}\sum_{m=-\infty}^{\infty}(e^{i2\delta_m} - e^{i2\delta_m^{\rm AB}})e^{im\theta}. \quad (18b)$$

Note that the phase distortion in the incident wave remains the same as in the pure AB case. These results are also valid for other scalar potential $V(r)$ that is vanishing beyond $r = a$. The subsequent task is to find the phase shifts δ_m.

Now we deal with the first process where $A(r)$ is given by equation (3a). In this case

$$R_m(r) = A_m J_{|m-\nu|}(kr), \quad r < a, \quad (19)$$

where the constant A_m can be determined in terms of a_m by the connection condition. It can be shown that the phase shifts in this case are determined by

$$\tan\left(\delta_m - \frac{\nu\pi}{2}\right) = \frac{C_1 - \Omega J_{m-\nu}(\xi)J_{|m-\nu|}(\xi)}{C_2 - \Omega N_{m-\nu}(\xi)J_{|m-\nu|}(\xi)}, \quad (20a)$$

where $\xi = ka$, and

$$C_1 = \frac{2}{\pi}\sin\frac{m-\nu-|m-\nu|}{2}\pi, \quad C_2 = \frac{2}{\pi}\cos\frac{m-\nu-|m-\nu|}{2}\pi. \quad (20b)$$

When $\Omega = 0$ this yields the result (12). On the other hand, when $\Omega \to \infty$ this gives the result for the case of an AB potential plus a hard core potential. It can be found that

see equation (21) above

$$\Delta f(\theta)=\sqrt{\frac{2}{\pi k}}\sin\nu\pi e^{im_0\theta}\left\{\sum_{n=0}^{\infty}\frac{e^{-i\alpha\pi}J_{n+\alpha}^2(\xi)e^{in\theta}}{[(-)^nJ_{n+\alpha}(\xi)J_{-n-\alpha}(\xi)+(2/\pi\Omega)\sin\alpha\pi]-e^{-i\alpha\pi}J_{n+\alpha}^2(\xi)}\right.$$
$$\left.+\sum_{n=0}^{\infty}\frac{e^{i\alpha\pi}J_{n+1-\alpha}^2(\xi)e^{-i(n+1)\theta}}{[(-)^nJ_{n+1-\alpha}(\xi)J_{-n-1+\alpha}(\xi)+(2/\pi\Omega)\sin\alpha\pi]+e^{i\alpha\pi}J_{n+1-\alpha}^2(\xi)}\right\}. \quad (23)$$

When $\nu=m_0$ (or $\alpha=0$), this immediately reduces to

$$\Delta f(\theta)=e^{im_0(\theta+\pi)}f_{\rm cr}(\theta), \quad (22a)$$

where

$$f_{\rm cr}(\theta)=i\sqrt{\frac{2}{\pi k}}\sum_{m=-\infty}^{\infty}\frac{J_m^2(\xi)e^{im\theta}}{J_m(\xi)H_m^{(1)}(\xi)-2i/\pi\Omega} \quad (22b)$$

is the scattering amplitude for a pure circular ring potential. In this case $f_{\rm AB}(\theta)=0$, so we have

$$f(\theta)=e^{im_0(\theta+\pi)}f_{\rm cr}(\theta), \quad (22c)$$

and the differential scattering cross-section is

$$\sigma(\theta)=\sigma_{\rm cr}(\theta)=|f_{\rm cr}(\theta)|^2. \quad (22d)$$

This means that the AB potential has no effect in this case as expected. It is not difficult to show that similar result also holds for other scalar potential $V(r)$ as long as $A(r)$ is given by equation (3a).

In the above we have obtained the exact scattering amplitude for the combination of the AB potential and the circular ring potential, given by equations (18a) and (21). The result is rather complicated. In the following we focus our attention on the limit case $a\to 0$ (then $\xi\to 0$). Since the result for $\alpha=0$ is simple and has been discussed above for general a, we only consider the case with $0<\alpha<1$. In this case we recast equation (21) in the following form that only involves the Bessel functions:

see equation (23) above

This is more suitable for approximate calculations. When $\xi\to 0$, every term has a finite denominator and an infinitesimal numerator, so the correction to the AB scattering amplitude is in general small. This is just what is expected and is trivial. However, when Ω takes some special values, we have finite corrections. First, when $\Omega=-2\alpha$, the $n=0$ term in the first sum gives a finite contribution. If infinitesimal terms are omitted, we have in this case

$$\Delta f(\theta)=-\sqrt{\frac{2}{\pi k}}\sin\nu\pi e^{im_0\theta},$$
$$f(\theta)=i\frac{\sin\nu\pi}{\sqrt{2\pi k}}\frac{e^{i(m_0+1/2)\theta}}{\sin(\theta/2)}=e^{i\theta}f_{\rm AB}(\theta). \quad (24a)$$

Second, when $\Omega=-2(1-\alpha)$, the $n=0$ term in the second sum gives a finite contribution, and we have

$$\Delta f(\theta)=\sqrt{\frac{2}{\pi k}}\sin\nu\pi e^{i(m_0-1)\theta},$$
$$f(\theta)=i\frac{\sin\nu\pi}{\sqrt{2\pi k}}\frac{e^{i(m_0-3/2)\theta}}{\sin(\theta/2)}=e^{-i\theta}f_{\rm AB}(\theta). \quad (24b)$$

In both cases we have finite corrections to the scattering amplitude, but the scattering cross-section remains unchanged. However, when the two conditions are satisfied simultaneously, which requires

$$\alpha=\tfrac{1}{2},\quad \Omega=-1, \quad (25)$$

then both terms are present in $\Delta f(\theta)$, and the scattering amplitude and cross-section turn out to be

$$f(\theta)=(2\cos\theta-1)f_{\rm AB}(\theta),$$
$$\sigma(\theta)=(2\cos\theta-1)^2\sigma_{\rm AB}(\theta). \quad (26)$$

In particular, we have $\sigma(\pi/3)=0$ and $\sigma(\pi)=9\sigma_{\rm AB}(\pi)$, so the backward scattering is remarkably amplified by the contact potential. In conclusion, when the magnetic flux and the strength of the contact potential take special values, the AB scattering cross-section is manifestly changed. This seems to be unexpected.

Let us have a further look into the above result. First we examine the phase shifts determined by equation (20). It can be shown that when $a\to 0$ the phase shifts in general tend to those for pure AB scattering given in equation (12), this is an expected result. However, under the condition (25), the situation is subtle and two of them have unusual limits:

$$\delta_{m_0}\to m_0\pi-\frac{\nu\pi}{2},\quad \delta_{m_0-1}\to\frac{\nu\pi}{2}. \quad (27)$$

Second we examine the bound states and see what happens in this case. Bound states with negative energy $E<0$ may be expected when $\Omega<0$. We denote $\kappa=\sqrt{2M|E|}/\hbar$ and the wave function of bound state in the mth angular momentum channel as $\psi_m^{\rm b}(r,\theta)=R_m^{\rm b}(r)e^{im\theta}$, then the radial wave function satisfies

$$\frac{d^2R_m^{\rm b}}{dr^2}+\frac{1}{r}\frac{dR_m^{\rm b}}{dr}-\left[\kappa^2+\frac{(m-\nu)^2}{r^2}+\frac{\Omega}{a}\delta(r-a)\right]R_m^{\rm b}=0. \quad (28)$$

The solution of this equation is

$$R_m^{\rm b}(r)=\begin{cases}b_mI_{|m-\nu|}(\kappa r), & r<a,\\ c_mK_{|m-\nu|}(\kappa r), & r>a,\end{cases} \quad (29)$$

where $I_{|m-\nu|}(\kappa r)$ and $K_{|m-\nu|}(\kappa r)$ are Bessel functions of imaginary argument, and the coefficients b_m and c_m are determined by the connection and normalization conditions. The connection condition (7) leads to the transcendental equation for the energy levels:

$$I_{|m-\nu|}(\kappa a)K_{|m-\nu|}(\kappa a)=-1/\Omega. \quad (30)$$

Since the left-hand side is positive, there may exist some root for this equation only when $\Omega < 0$ as anticipated above. From the graph it can be seen that the function $I_{|m-\nu|}(\kappa a)K_{|m-\nu|}(\kappa a)$ decreases monotonically when the argument increases. Thus there is one (and only one) root for the above equation when $1/|\Omega| \leq I_{|m-\nu|}(x)K_{|m-\nu|}(x)|_{x=0}$, or

$$|\Omega| \geq 2|m-\nu| = 2|m-m_0+\alpha|. \tag{31}$$

Therefore, we have a bound state in the mth channel when the above condition holds with the larger sign, or a half-bound state [20] when it holds with the equal sign. With a larger Ω, we have bound (or half-bound) states in more channels. Under the condition (25), we have two half-bound states, one with $m = m_0$ and the other with $m = m_0 - 1$, and no bound or half-bound states in other channels. Therefore, the unusual result for the phase shifts in equation (27) is associated with the existence of half-bound states in the same channels, which in turn leads to the unexpected scattering cross-section in equation (26) when $a \to 0$. This is similar to the situation for a pure contact potential in three and one dimensions [19].

From equation (20) we see that the phase shifts depend only on $\xi = ka$, rather than in k and a separately. Thus $\lim_{a\to 0}\delta_m$ is the same as the threshold phase shift $\lim_{k\to 0}\delta_m$. In the study of the Levinson theorem it has been known for a long time that the existence of a half-bound state in some angular momentum channel may lead to an unusual result for the threshold phase shift in the same channel [20,29,30]. Therefore the above result is not quite surprising.

Now we deal with the second process where $A(r)$ is given by equation (3b). In this case

$$R_m(r) = B_m r^{|m|}\exp\left(-\frac{\nu r^2}{2a^2}\right)F\left(\frac{|m|-m+1}{2} -\frac{\xi^2}{4\nu}, |m|+1, \frac{\nu r^2}{a^2}\right), \quad r<a, \tag{32}$$

where $F(\beta,\gamma,z)$ is the confluent hypergeometric function [31] and B_m is a constant. It can be shown that the phase shifts in this case are determined by

$$\tan\left(\delta_m - \frac{\nu\pi}{2}\right) = \frac{\xi J'_{m-\nu}(\xi)F_m(\xi) - J_{m-\nu}(\xi)G_m(\xi)}{\xi N'_{m-\nu}(\xi)F_m(\xi) - N_{m-\nu}(\xi)G_m(\xi)}, \tag{33}$$

where

$$F_m(\xi) = F\left(\frac{|m|-m+1}{2} - \frac{\xi^2}{4\nu}, |m|+1, \nu\right), \tag{34a}$$

$$G_m(\xi) = \left(m-\nu-1+\Omega+\frac{\xi^2}{2\nu}\right)F\left(\frac{|m|-m+1}{2} -\frac{\xi^2}{4\nu}, |m|+1, \nu\right) + \left(|m|-m+1-\frac{\xi^2}{2\nu}\right) \times F\left(\frac{|m|-m+3}{2} - \frac{\xi^2}{4\nu}, |m|+1, \nu\right). \tag{34b}$$

In obtaining these results the functional relations of the confluent hypergeometric functions [31] have been used.

If $\nu = m_0$ (or $\alpha = 0$), it turns out that $f(\theta) = 0$ in the limit $a \to 0$ as expected.

When $0 < \alpha < 1$, we have the result

$$\Delta f(\theta) = \sqrt{\frac{2}{\pi k}}\sin\nu\pi e^{im_0\theta}\left[\sum_{n=0}^{\infty}\frac{c_n(\xi)e^{in\theta}}{(-)^n e^{i\alpha\pi}C_n(\xi) - c_n(\xi)} + \sum_{n=0}^{\infty}\frac{d_n(\xi)e^{-i(n+1)\theta}}{(-)^n e^{-i\alpha\pi}D_n(\xi) + d_n(\xi)}\right], \tag{35a}$$

where

$$c_n(\xi) = \xi J'_{n+\alpha}(\xi)F_{m_0+n}(\xi) - J_{n+\alpha}(\xi)G_{m_0+n}(\xi), \tag{35b}$$

$$C_n(\xi) = \xi J'_{-n-\alpha}(\xi)F_{m_0+n}(\xi) - J_{-n-\alpha}(\xi)G_{m_0+n}(\xi), \tag{35c}$$

$$d_n(\xi) = \xi J'_{n+1-\alpha}(\xi)F_{m_0-n-1}(\xi) - J_{n+1-\alpha}(\xi)G_{m_0-n-1}(\xi), \tag{35d}$$

$$D_n(\xi) = \xi J'_{-n-1+\alpha}(\xi)F_{m_0-n-1}(\xi) - J_{-n-1+\alpha}(\xi)G_{m_0-n-1}(\xi). \tag{35e}$$

When $\xi \to 0$, every term has an infinite denominator and an infinitesimal numerator, so the correction to the AB scattering amplitude is in general negligible. This is also an expected result and is trivial. However, when Ω and α satisfy some relation, we have finite corrections. First, when $G_{m_0}(0)/F_{m_0}(0) = -\alpha$, the $n = 0$ term in the first sum gives a finite contribution and we have the result given in equation (24a). Second, when $G_{m_0-1}(0)/F_{m_0-1}(0) = -(1-\alpha)$, the $n = 0$ term in the second sum gives a finite contribution and we have the result given in equation (24b). In both cases the scattering cross-section remains unchanged as before. However, when the two conditions are satisfied simultaneously, the scattering cross section is changed and is given by equation (26). As before, both α and Ω are determined by the two conditions. If $m_0 = n \in \mathbb{N}$, α is determined by the transcendental equation

$$\frac{F(3/2,n,n-\alpha)}{F(1/2,n,n-\alpha)} - \frac{F(3/2,n+1,n-\alpha)}{F(1/2,n+1,n-\alpha)} - 2\alpha = 0, \tag{36}$$

and Ω is given by

$$\Omega = \Omega(\alpha) = 1 - \frac{F(3/2,n,n-\alpha)}{F(1/2,n,n-\alpha)}. \tag{37}$$

We denote the root of equation (36) as α_n, the corresponding value $\Omega(\alpha_n)$ as Ω_n, and the parameter $\nu = n - \alpha_n$ as ν_n. If $m_0 = 1 - n$ where $n \in \mathbb{N}$, α is determined by the transcendental equation

$$\frac{F(3/2,n,n-(1-\alpha))}{F(1/2,n,n-(1-\alpha))} - \frac{F(3/2,n+1,n-(1-\alpha))}{F(1/2,n+1,n-(1-\alpha))} - 2(1-\alpha) = 0, \tag{38}$$

and Ω is given by

$$\Omega = \Omega(\alpha) = 1 - \frac{F(3/2,n,n-(1-\alpha))}{F(1/2,n,n-(1-\alpha))}. \tag{39}$$

Table 1. Numerical solutions of equations (36) and (37).

n	1	2	3	4	5	10	20	30	40	50	100	1000	5000
α_n	0.226	0.244	0.250	0.254	0.256	0.261	0.264	0.266	0.267	0.267	0.268	0.271	0.271
Ω_n	−0.921	−1.34	−1.66	−1.92	−2.15	−3.05	−4.31	−5.28	−6.09	−6.80	−9.61	−30.3	−67.7

In obtaining equations (38) and (39) the functional relations of the confluent hypergeometric functions have been used. We denote the root of equation (38) as α_{1-n}, the corresponding value $\Omega(\alpha_{1-n})$ as Ω_{1-n}, and the parameter $\nu = 1 - n - \alpha_{1-n}$ as ν_{1-n}. Comparing equations (36) and (38), it is easy to find that $\alpha_{1-n} = 1 - \alpha_n$, and $\Omega_{1-n} = \Omega_n$, so that $\nu_{1-n} = -\nu_n$. Therefore we conclude that when

$$\nu = \pm\nu_n = \pm(n - \alpha_n), \quad \Omega = \Omega_n, \quad n \in \mathbb{N}, \tag{40}$$

the AB scattering cross-section is changed by the contact potential and is given by equation (26). This result is essentially the same as that obtained in the first process. The difference is the dependence of α_n and Ω_n on n in the present case. The above result also shows that the cross-section does not depend on the sign of the magnetic flux just as expected.

The above discussions make sense only when equation (36) has a solution α in the open interval $(0, 1)$. It turns out that there is one and only one such solution for a rather wide range of n. Some numerical results are listed in Table 1. It can be seen that α_n varies with n very slowly, and for small n, Ω_n is close to the result $\Omega = -1$ obtained in the first process.

To understand the above result we again examine the phase shifts and bound states in the present case. Now the phase shifts are determined by equation (33). As before we find that they in general tend to those for pure AB scattering when $a \to 0$. However, when the condition (40) is fulfilled, two of them tend to the unusual result given by equation (27). As for the bound states, the radial wave equation in the present case reads

$$\frac{d^2R_m^{\mathrm{b}}}{dr^2} + \frac{1}{r}\frac{dR_m^{\mathrm{b}}}{dr} - \left[\kappa^2 + \left(\frac{m}{r} - \frac{qA}{\hbar c}\right)^2 + \frac{\Omega}{a}\delta(r-a)\right] R_m^{\mathrm{b}} = 0, \tag{41}$$

where $A(r)$ is given by equation (3b). The solution of this equation is

$$R_m^{\mathrm{b}}(r) = \begin{cases} b_m r^{|m|} \exp\left(-\dfrac{\nu r^2}{2a^2}\right) F\left(\dfrac{|m| - m + 1}{2} + \dfrac{\kappa^2 a^2}{4\nu}, |m| + 1, \dfrac{\nu r^2}{a^2}\right), & r < a, \\ c_m K_{|m-\nu|}(\kappa r), & r > a. \end{cases} \tag{42}$$

The connection condition (7) leads to the transcendental equation for the energy levels:

$$\Omega = \nu - |m| + \frac{\kappa a K'_{|m-\nu|}(\kappa a)}{K_{|m-\nu|}(\kappa a)} - 2\nu \frac{F'[(|m| - m + 1)/2 + \kappa^2 a^2/4\nu, |m| + 1, \nu]}{F[(|m| - m + 1)/2 + \kappa^2 a^2/4\nu, |m| + 1, \nu]}. \tag{43}$$

This is rather complicated. We will not study it in detail, but only consider some half-bound states. The condition for a half-bound state in the mth channel is

$$\Omega = 1 - |m - \nu| - (m - \nu) - (|m| - m + 1)\frac{F[(|m| - m + 3)/2, |m| + 1, \nu]}{F[(|m| - m + 1)/2, |m| + 1, \nu]}. \tag{44}$$

This is still complicated. However, it is not difficult to show that the condition for the simultaneous existence of two half-bound states, one with $m = m_0$ and the other with $m = m_0 - 1$, is equation (40). Thus we see again that the unexpected result for the scattering cross-section is associated with the existence of two half-bound states.

In summary we have studied the effect of a contact potential on the AB scattering. It turns out that when the magnetic flux and the strength of the contact potential take some special values, the AB scattering cross-section is manifestly changed. This result appears to be unexpected. We have shown that this unexpected result is associated with the existence of two half-bound states in two adjacent angular momentum channels, which yield unusual result for the phase shifts in the corresponding channels. To deal with the singularity of the contact potential, two different limiting processes are presented, and the results obtained are qualitatively the same in the two processes.

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